On weak coupling expansion in models with unbroken symmetry * †

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An investigation of the weak coupling region of $2D$ $SU(N)$ spin models is presented. An expansion of the free energy and correlation functions at low temperatures is performed in the link formulation with periodic boundary conditions (BC). The resulting asymptotics is shown to be nonuniform in the volume for the free energy.

This paper deals with $2D$ $SU(N) \times SU(N)$ principal chiral model in the weak coupling region whose partition function (PF) is given by

$$Z = \int \prod_x DU_x \exp\left[\beta \sum_{x,n} \text{ Re Tr} U_x U^+_{x+n}\right], \quad (1)$$

where $U_x \in SU(N)$, DU_x is the invariant measure and we assume periodic BC. The main question discussed here is what is the correct asymptotic expansion of nonabelian models at large β and whether the conventional perturbation theory (PT) gives the correct expansion, the question addressed few years ago in [1]. As was rigorously proven, the conventional PT gives an asymptotic expansion which is uniform in the volume for the abelian XY model [2]. One of the basic theorem which underlies the proof states that the following inequality holds in the $3D$ XY model

$$< \exp(\sqrt{\beta} A(\phi_x)) > \leq \ C, \quad (2)$$

where C is β-independent and $A(\phi + 2\pi) = \phi$. ϕ_x is an angle parametrizing the action of the XY model $S = \sum_{x,n} \cos(\phi_x - \phi_{x+n})$. It follows, at large β the Gibbs measure is concentrated around $\phi_x \approx 0$ giving a possibility to construct an expansion around $\phi_x = 0$. This inequality is not true in $2D$ because of the Mermin-Wagner theorem (MWT), however the authors of [2] prove the same inequality for the link angle, i.e.

$$< \exp(\sqrt{\beta} A(\phi_l)) > \leq \ C \ , \phi_l = \phi_x - \phi_{x+n}. \quad (3)$$

Thus, in $2D$ the Gibbs measure at large β is concentrated around $\phi_l \approx 0$ and the asymptotics can be constructed expanding in powers of ϕ_l. In the abelian case such an expansion is equivalent to the expansion around $\phi_x = 0$ because i) the action depends only on the difference $\phi_x - \phi_{x+n}$ and ii) the integration measure is flat, $DU_x = d\phi_x$.

I'm not aware of any similar rigorous result proven for the nonabelian model. Nevertheless, because of the MWT one has to expect something similar to (3), namely

$$< \exp(\sqrt{\beta} \text{arg} A(\text{Tr} V_l)) > \leq C \ , V_l = U_x U^+_{x+n}. \quad (4)$$

This is equivalent to the statement that the Gibbs measure at large β concentrates around $V_l \approx I$. That such (or similar) inequality holds in $2D$ models is intuitively clear and should also follow from chessboard estimates. It is assumed here that (4) is correct, hence $V_l = I$ is the only saddle point for the invariant integrals. Thus, correct asymptotic expansion should be given via expansion around $V_l = I$, similarly to the abelian model. If the conventional PT gives correct expansion, it must reproduce the series obtained by expansion around $V_l = I$. However, neither i) nor ii) holds in the nonabelian models, therefore it is far from obvious that two expansions indeed coincide. Our purpose here is to develop an expansion around $V_l = I$ aiming to calculate the asymptotic series for nonabelian models.

To accomplish this task we make a change of variables $V_l = U_x U^+_{x+n}$ in (1). PF becomes [3]

$$Z = \int \prod_l dV_l \exp\left[\beta \sum_l \text{Re Tr} V_l + \ln J(V)\right], \quad (5)$$

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$$J(V)=\prod_p\left[\sum_r d_r\chi_r\left(\prod_{l\in p}V_l\right)\right]. \qquad (6)$$

$\prod_p$ is a product over all plaquettes of $2D$ lattice, the sum over r is sum over all representations of $SU(N)$, $d_r=\chi_r(I)$ and

$$\prod_{l\in p}V_l=V_n(x)V_m(x+n)V_n^+(x+m)V_m^+(x). \qquad (7)$$

$J(V)$ is a product of $SU(N)$ delta-functions which introduce the constraint $\prod_{l\in p}V_l=I$ [3]. Its solution is a pure gauge $V_l=U_xU_{x+n}^+$, thus two forms of the PF are equivalent. The correlation function $\Gamma(x,y)=<\mathrm{Tr}U_xU_y^+>$ becomes

$$\Gamma(x,y)=<\mathrm{Tr}\prod_{l\in C_{xy}}W_l>, \qquad (8)$$

where C_{xy} is some path connecting points x and y and $W_l=V_l$ if along the path C_{xy} the link l goes in the positive direction and $W_l=V_l^+$, otherwise. The abelian analog of (5) reads

$$Z_{XY}=\int\prod_l\left[d\phi_l e^{\beta\cos\phi_l}\right]\prod_p\sum_{r=-\infty}^{\infty}e^{ir\phi_p}, \qquad (9)$$

$\phi_p=\phi_n(x)+\phi_m(x+n)-\phi_n(x+m)-\phi_m(x+n)$.

The main step is to find a form for (5) appropriate for large-β expansion. We consider $SU(2)$ model and parametrize (σ^k are Pauli matrices)

$$V_l=\exp[i\sigma^k\omega_k(l)]\ ,\ k=1,2,3\ , \qquad (10)$$

$$W_l=\left[\sum_k\omega_k^2(l)\right]^{1/2},\ W_p=\left[\sum_k\omega_k^2(p)\right]^{1/2}, \qquad (11)$$

where $\omega_k(p)$ is a plaquette angle defined as

$$V_p=\prod_{l\in p}V_l=\exp[i\sigma^k\omega_k(p)]. \qquad (12)$$

Then, the PF (5) can be exactly rewritten to the following form on a dual lattice ($p\rightarrow x$)

$$Z=\int\prod_l\left[\frac{\sin^2W_l}{W_l^2}e^{\beta\cos W_l}\prod_k d\omega_k(l)\right]$$

$$\prod_x\frac{W_x}{\sin W_x}\prod_x\sum_{m(x)=-\infty}^{\infty}\int\prod_k d\alpha_k(x)$$

$$\exp\left[-i\sum_k\alpha_k(x)\omega_k(x)+2\pi im(x)\alpha(x)\right]\ , \qquad (13)$$

where we have introduced the auxiliary field $\alpha_k(x)$ and $\alpha(x)=(\sum_k\alpha_k^2(x))^{1/2}$. The derivation of the PF (13) is given in [4]. To perform large-β expansion we proceed in a standard way, i.e. first we make the substitution

$$\omega_k(l)\rightarrow(\beta)^{-1/2}\omega_k(l)\ ,\ \alpha_k(x)\rightarrow(\beta)^{1/2}\alpha_k(x) \qquad (14)$$

and then expand in powers of fluctuations of link fields. Technical details are left to a forthcoming paper [4]. The essential aspects are:

1). The main building blocks of the expansion are the following functions which we term "link" Green functions

$$G_{ll'}=2\delta_{l,l'}-G_{x,x'}-G_{x+n,x'+n'}+G_{x,x'+n'}$$
$$+G_{x+n,x'},\ D_l(x')=G_{x,x'}-G_{x+n,x'}. \qquad (15)$$

$G_{x,x'}$ is a "standard" Green function on the periodic lattice. Unlike $G_{x,x'}$, both $G_{ll'}$ and $D_l(x')$ are infrared finite by construction.

2). Sums over representations are treated via the Poisson resummation formula. Due to this, zero modes decouple from the expansion after integration over the auxiliary field.

3). An ensemble of fluctuations around $V_l=I$ can be written as usual Gaussian ensemble appearing due to integration over representations (over auxiliary field) in abelian (nonabelian) case.

4). Generating functionals (GF) appear to be:

$$M(h_l)=\exp\left[\ \frac{1}{4}h_lG_{ll'}h_{l'}\ \right],\ G_{ll}=1 \qquad (16)$$

for the XY model and

$$M(h_l,s_x)=\exp[\ \frac{1}{4}s_k(x)G_{x,x'}s_k(x')$$
$$-\frac{i}{2}s_k(x)D_l(x)h_k(l)+\frac{1}{4}h_k(l)G_{ll'}h_k(l')\ ]\ , \qquad (17)$$

for the $SU(2)$ model. $h_k(l)$ ($s_k(x)$) is a source for the link (auxiliary) field and sum over all repeating indices is understood. We list below some results proven in [4].

[3]On the periodic lattice one has to constraint two holonomy operators, i.e. closed paths winding around the whole lattice. Such global constraints do not influence thermodynamic limit in $2D$.

I. Asymptotics for the free energy of the XY model can be obtained from the expansion

$$Z_{XY} \sim \prod_l \left[1 + \sum_{k=1}^{\infty} \frac{1}{\beta^k} A_k (\frac{\partial^2}{\partial h_l^2}) \right] M(h_l) \ , \qquad (18)$$

where A_k are known coefficients. It is easy to prove that this expansion coincides with the standard PT answer. For example, the first order coefficient is given by

$$\frac{1}{L^2} C^1_{XY} = \frac{1}{32L^2} \sum_l G^2_{ll} = \frac{1}{16} \ . \qquad (19)$$

II. Correlation function (8) in the $SU(2)$ model up to the first order is given by

$$\Gamma(x,y) = 1 - \frac{3}{4\beta} \sum_{l,l' \in C^d_{xy}} G_{ll'} + O(\beta^{-2}) \ , \qquad (20)$$

where C^d_{xy} is a path dual to the path C_{xy}, i.e consisting of the dual links which are orthogonal to the original links $l, l' \in C_{xy}$. The form of $G_{ll'}$ guarantees independence of $\Gamma(x,y)$ of the choice of the path C_{xy}. After some algebra it is easy to recover the standard PT answer

$$\Gamma(x,y) = 1 - \frac{3}{2\beta} D(x-y) \ , \qquad (21)$$

$$D(x) = \frac{1}{L^2} \sum_{k_n=1}^{L-1} \frac{1 - e^{\frac{2\pi i}{L} k_n x_n}}{D - \sum_{n=1}^{D} \cos \frac{2\pi}{L} k_n} \ .$$

III. The main result of this study is the first order coefficient of the $SU(2)$ free energy. The second order term in the auxiliary fields gives the following contribution at this order

$$\frac{1}{L^2} C^1_{div} = \frac{3}{8L^2} \sum_{x,x'} G_{x,x'} Q_{x,x'} \ . \qquad (22)$$

$Q_{x,x'}$ is a second order polynomial in link Green functions (15). Numerical study of (22) shows that $\frac{1}{L^2} C^1_{div} = b + a_1 \ln L + O(\frac{\ln L}{L})$. All other contributions coming from the action, from the measure and from $J(V)$ depend only on the link Green functions and are finite. The final result is

$$\frac{1}{L^2} C^1_{SU(2)} = a_0 + a_1 \ln L + O(\frac{\ln L}{L}) \ , \qquad (23)$$

where a_i are β- and L-independent.

Conclusions: The GF of the XY model depends only on $G_{ll'}$, guaranteeing the infrared finitness of the expansion. This is a direct consequence of the fact that the action of XY model is a function of ϕ_l. It is not the case in nonabelian models: the GF (17) includes also dependence on $G_{x,x'}$ creating potential danger for nonuniformity. The first order coefficient of $\Gamma(x,y)$ is expressed only via $G_{ll'}$ recovering thus the standard PT result. A genuine nonabelian contribution appears only in the first order coefficient of the free enegy or in the second order coefficient of $\Gamma(x,y)$. That there might be a problem with 2-nd order coefficient of $\Gamma(x,y)$ has been shown in [1], namely the PT with superinstanton BC (SIBC) gives a result different from the PT result obtained with periodic or Dirichlet BC. As was shown later, the PT with SIBC diverges at the third order [5]. The interpretation has been given that it is just SIBC for which the asymptotics is nonuniform in the volume. Our result seems to support other interpretation: the correct asymptotic expansion of $2D$ nonabelian models is nonuniform in the volume. In particular, it means that the conventional PT does not take into account all the saddle points when the volume is increasing and becomes larger than the perturbative correlation length. We conclude thus either 1) to get the correct expansion one has to expand a true function in the thermodynamic limit (like in $1D$) or 2) the inequality (4) does not hold, e.g. the factor $\sqrt{\beta}$ is incorrect or 3) nonabelian models are not expandable in $1/\beta$ in two dimensions.

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